# MATHEMATICAL MODELS.

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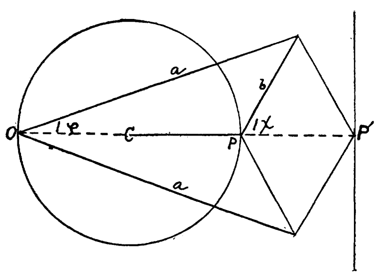
By an ordinary consideration of geometrical forms it might appear that a mathematical model could have but a theoretical interest, and non-mathematicians are mostly liable to the opinion that solid representations in space be rather curiosities than practical schemes. This opinion was prevailing even among mathematicians until some 20 years ago, when besides of the analytical results particular stress was also laid upon the conception of the geometrical forms. It was perceived that real progress in mathematics could only be made by aid of geometrical illustrations, and since that time the geometrical standpoint—geometry taken in the most general sense—is dominant in mathematical investigation. At present it is not sufficient to know and to discuss the analytical equation of a geometrical form; its real shape must be studied also. Thus, if a plane configuration, or a representation by descriptive geometry is too complicated, or not conspicuous enough, a model of the form is constructed. This enables the student or the investigator to see the essential features of the form, and suggests to him new ideas. Moreover, it is obvious that by the constant reference to real forms many problems of technics come into account which by a purely analytical method, with the exclusion of every configuration, never would be taken up. In this manner the very important connection between science and technics can be maintained, and there is no danger that mathematics will branch off too much from its technical application.

A simple illustration for this method is, for instance, Peoncellier's diagram, which transforms a circular movement into a movement of a straight line. Peoncellier published a solution of this problem in the "Nouvelles Annales de Mathematic," in 1864. The main part of the apparatus for the realization of this movement on a straight line is called "inversor," because it produces the relation of the inversion.

The "inversor" consists of two bars of equal length which are connected in the point O (see figure), and between which a system of other bars in shape of a rhombus is put in. At all connection points of the bars are links, so that the whole system is moveable. If the point O is kept fixed, and if one point of the rhombus, P, moves at pleasure, the other point, P′, will move such that it is inverse to P, or $OP \times OP' = \text{const.}$ For, by using the designations of the figures, the relations exist:




$$
\begin{aligned}
a \sin e &= b \sin x, \text{ or} \\
O &= a^2 \sin^2 e - b^2 \sin^2 x \\
OP &= a \cos e - b \cos x \\
OP' &= a \cos e + b \cos x \\
\hline
OP \times OP' &= a^2 \cos^2 e - b^2 \cos^2 x \\
\text{adding } O &= a^2 \sin^2 e - b^2 \sin^2 x \\
\hline
OP \times OP' &= a^2 - b^2 = \text{const.}
\end{aligned}
$$

Thus, the inversor realizes the transformation by reciprocal radii in regard to

a circle with O as a center and $\sqrt{a^2 b^2}$ as a radius. Now it is easy to produce a straight line by P′. To perform this it is only necessary to move the point P on a circle passing through O. This can be done by connecting the center C of the circle with the point P by a seventh bar C P.

As to mathematical means of illustrations in general, and their utility, it may be well to mention some points of Felix Klein's lectures on mathematics, on the occasion of the World's Fair, in Chicago. (Felix Klein, of Gottingen, at present one of the most eminent mathematicians, and well known to American universities, delivered those lectures before the members of the Congress of Mathematics, at Northwestern University, Evanston, Ill.)

Among mathematicians in general three main categories may be distinguished, and perhaps the names logicians, formalists and intuitionists may serve to characterize them. (1) The word logician is here only intended to indicate that the main strength of the men belonging to this class lies in their logical and critical power, in their ability to give strict definitions, and to derive rigid deductions therefrom. (2) The formalists among the mathematicians excel mainly in the skilful formal treatment of a given question, in devising for it an "algorithm." (3) To the intuitionists, finally, belong those who lay particular stress on geometrical intuition, not in pure geometry only, but in all branches of mathematics. What Benjamin Pearce has called "geometrizing a mathematical question" seems to express the same idea.

Klein ranks himself to the logicians and intuitionists, and possesses the power and ability of both. The main feature of this combination lies in the "refined intuition." Refined intuition characterizes the most successful mathematical schools of the present, and does not properly mean intuition as it is required for an artisan or a draughtsman. The latter, or the naive intuition, is not exact, while the refined intuition is not properly intuition at all, but arises through the logical development from axioms considered as perfectly exact. Not all mathematicians have this point of view, what might be explained by the fact that the degree of exactness of the intuition of space is different in different individuals,—perhaps in different races. Klein points out, as if a strong naive space-intuition were an attribute preeminently of the Teutonic race, while the critical, purely logical sense is more fully developed in the Latin and Hebrew races.

But in general the henristic value of the applied sciences as an aid to discover new truths in mathematics is at present more recognized than ever before. Besides the great importance in technics on other applied sciences of mathematics, the henristic method is the most successful in pedagogics. An education which develops in the student a strong intuition produces, as the history of pedagogics shows it, the best results. It was one of the first principles of Pestalozzi, and is useful as well in the higher as in the common schools. The incomparable success of the lectures of the great mathematician Steiner was due to this method. Steiner himself had such an unusual intuition of space that it was easy for him to make the most difficult constructions of descriptive geometry by imagination, and to see clearly before his powerful mind complicated geometrical configurations. That a strong intuition is of a general value may be stated by the same Steiner. Whenever he visited an art exhibition artists and experts were surprised by his quick and correct criticism. It took him only a glance on a person to not forget her image any more.

People generally believe that intuition, like fine arts, must be born with a person. This may be true in a certain sense, and only to a certain limit.

Education and training, however, can produce a good deal of what is called ability and art in common life. I shall give an example which may suggest a test for the quality of the naive intuition of a person. Most of the people that look at a tree think it to be a very simple object of art. If they would shut their eyes at the moment of this intuition, and try to imagine exactly the tree which they just were going to see, they would find out how strong their intuition was. Even representants of the purely analytical school would be surprised after this test. Thus, it is the aim of the leading mathematicians of the present to proceed from the naive intuition to the refined intuition, i. e., to combine the critical power of logic with the usefulness of intuition.

When I was in Zurich, an eminent mathematician, who is now in Germany, lectured on the theory of functions; but his treatment of the subject was purely analytical. One day a student showed him a short geometrical demonstration of one of his theorems. The professor looked at it, and said: "This seems to be a very elegant prove, indeed, but I am sorry to say that I cannot understand it—I am not a geometrician." However valuable those lectures have been, such a one-sided standpoint cannot be taken any more. One-sidedness must disappear.

The question now arises as to how produce a stronger intuition. It has been answered many times with more or less success, and by the most prominent educators. According to their statements, the only way is a thorough course in what is called mathematical graphics.

From what has been said before it will be understood that we define mathematical models in the most general sense, i. e., as geometrical figures and configurations, and mathematical models in a narrower sense as geometrical and mechanical models. All these objects serve the one educational purpose to refine the intuition, and thereby the efficiency of applied sciences, or technics. Now I will proceed to what might appear first from the title of the paper, namely, the geometrical models or geometrical means of illustration.

In the accounts of the investigations of Euler, Newton and Cramer we find numerous figures. The interest for the construction of models was first produced in France, where, under the influence and the example of Monge, a great number of thread models of surfaces of the second order, conoids and helicoids were being constructed. A further progress was made by Bardin (1855). He had made cast models of stone-cuts, gearings, and many other things. His collection was greatly enlarged by Muret. These schemes were not appreciated by French mathematicians, while Cayley and Henrici (1876) exhibited in London mathematical models besides other scientific apparatus of the universities of Cambridge and London. In Germany the construction of models received an impetus, but after the assimilation of projective by descriptive geometry. Plucker drew already in 1835 the shapes of the curves of the third order, and made in 1868 the first larger collection of models, consisting of models of complex surfaces of the fourth order. Klein added to it some more models of the same character. A special surface of the fourth order, the wave surface of crystals with two axes, was constructed by Magnus in Berlin, and Soleil in Paris, in 1840. Fiedler, in Zurich, is the constructor of the first model of a surface of the third order with its 27 straight lines. He constructed it in 1868, when he was a professor in Prague, and always used to have it suspended at the ceiling of his study or parlor. Now this model is a historical curiosity of the mathematical collec-

tion of the Polytechnic of Zurich. Models of surfaces of the fourth order were made by Kummer from 1860 to 1870. His pupil-disciple, Schwarz, constructed also a set of models, among which are minimal surfaces and the central surface of the ellipsoid. On the occasion of a meeting of mathematicians in Gottingen, a large exhibition of models was arranged, which had a great success. Thus, A. Brill, F. Klein and W. Dyck started the construction of models in the mathematical seminary of the Polytechnic Institute of Munich in a systematic manner. Since 1877 more than a hundred models of all kinds have appeared there. They are not all intended purely for mathematical teaching, but also for the use in lectures on perspective, mechanics, and mathematical physics. As it is seen from this fact, higher institutions of mathematical learning begin to put particular stress upon the collection of models as means of illustration. The value of mathematical models is almost inappreciable, because there are no other means that contribute more to a refined intuition. Collections of mathematical models will become, for the mathematical departments of the universities, polytechnics, and colleges, what the museums of natural history are for the biological departments.